Forecast Uncertainty, Disagreement, and the Linear Pool

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*The views expressed in this presentation are those of the authors and do not necessarily reflect the positions or policies of the Deutsche Bundesbank
Introduction

- The linear prediction pool (LP) is widely used for density forecast combinations.
- If the individual density forecasts are variance-unbiased (‘neutrally dispersed’), the LP has a too large variance (is ‘overdispersed’, Gneiting & Ranjan 2013).
- We sharpen results concerning the variance bias of the LP.
- The centered linear pool (CLP) emerges as a natural (partial) fix, removing the “disagreement” component of the LP.
- We show that disagreement is uncorrelated with the squared mean forecast errors of the LP under conditions that can be empirically relevant.
A simple example

- Consider a DGP given by

\[ Y = M_1 + M_2 + U \]

with \( M_1 \sim N(0, 1), M_2 \sim N(0, 1), U \sim N(0, 1) \), all i.i.d.

- Two forecasters produce density forecasts. 1st forecaster only observes \( M_1 \), 2nd forecaster only observes \( M_2 \). Their ideal forecasts are

\[ f_i(Y) = \frac{1}{\sqrt{2\pi V_i}} \exp \left( -\frac{1}{2} \frac{(Y - M_i)^2}{V_i} \right) \]

with \( V_1 = V_2 = 2 \)

- LP combines these density forecasts as

\[ f_{LP}(Y) = \omega_1 f_1(Y) + (1 - \omega_1) f_2(Y) \]

with \( 0 \leq \omega_1 \leq 1 \)
Decomposition of the linear pool’s variance forecast

Variance of LP’s density can be decomposed into weighted individual variance forecasts ($\omega'V$) and disagreement ($D$) (e.g. Wallis, 2005)

$$V_{LP} = \sum_{i=1}^{n} \omega_i V_i + \sum_{i=1}^{n} \omega_i (M_i - M_c)^2,$$

where $n$ is number of forecasts and

$$M_c = \sum_{i=1}^{n} \omega_i M_i = \omega' M$$

is the combined mean forecast. Taking expectations gives

$$\mathbb{E} [V_{LP}] = \mathbb{E} [\omega'V] + \mathbb{E} [D]$$
Decomposition of the linear pool’s squared forecast errors

Denote forecaster $i$’s squared mean forecast error by $S_i = (Y - M_i)^2$, then

$$
\sum_{i=1}^{n} \omega_i S_i = \sum_{i=1}^{n} \omega_i (Y - M_i)^2
$$

$$
= (Y - M_c)^2 + \sum_{i=1}^{n} \omega_i (M_i - M_c)^2
$$

$$
= S + D
$$

where $S$ denotes squared forecast error of $M_c$

If forecasters make *unbiased variance forecasts*, i.e. if assumption

$$
\mathbb{E} [V_i] = \mathbb{E} [S_i]
$$

holds, this implies

$$
\mathbb{E} [S] = \mathbb{E} [\omega' V] - \mathbb{E} [D]
$$
Variance bias of the linear pool

Expected variance of LP equals

\[ \mathbb{E} [V_{LP}] = \mathbb{E} [\omega' V] + \mathbb{E} [D] \]

Expected squared mean forecast error of LP equals

\[ \mathbb{E} [S] = \mathbb{E} [\omega' V] - \mathbb{E} [D] \]

Thus, variance forecast of LP is upward biased by \( 2 \times \mathbb{E} [D] \), because

\[ \mathbb{E} [V_{LP}] = \mathbb{E} [S] + 2 \times \mathbb{E} [D] \]

\[ \rightarrow \] Removing disagreement component from LP reduces variance bias
A simple example cont’d

- Previous consideration suggests using centered linear pool (CLP)

\[ f_{\text{CLP}}(Y) = \omega_1 f_1^c(Y) + (1 - \omega_1) f_2^c(Y) \]

with

\[ f_i^c(Y) = \frac{1}{\sqrt{2\pi V_i}} \exp \left( -\frac{1}{2} \frac{(Y - M_c)^2}{V_i} \right) \]

and \( V_1 = V_2 = 2 \). This eliminates \( D \), giving

\[ V_{\text{CLP}} = \omega'V \Rightarrow \mathbb{E}[V_{\text{CLP}}] = \mathbb{E}[\omega'V] \]

- If one knew \( \mathbb{E}[D] \) or \( \mathbb{E}[(Y - M_c)^2] \), one could set up a variance-unbiased linear pool (VULP). For example, with \( \omega_1 = 0.5 \)

\[ f_{\text{VULP}}(Y) = 0.5 \tilde{f}_1(Y) + 0.5 \tilde{f}_2(Y) \]

with

\[ \tilde{f}_i(Y) = \frac{1}{\sqrt{2\pi \times 1.5}} \exp \left( -\frac{1}{2} \frac{(Y - M_c)^2}{1.5} \right) \]
A simple example cont’d

Forecast densities with $M_1 = -1$, $M_2 = 1$ and $\omega_1 = 0.5$
Correlation of disagreement and squared errors

- Disagreement $D$ might be helpful to predict squared forecast error $S$ of LP despite of bias, if $D$ and $S$ are sufficiently correlated
- If the assumptions
  - vector $(M', Y)'$ with $M = (M_1, M_2, \ldots, M_n)'$ follows multivariate normal distribution
  - mean forecasts $M_i$ are unbiased, i.e. $E[M_i] = E[Y]$ for all $i$
  - combination weights $\omega^*$ minimize squared mean forecast error and sum to 1 (Bates&Granger 1969)

  hold, then covariance of $D$ and $S$ equals

  $$\text{Cov}(D, S) = 0$$

- If all assumptions are fulfilled, disagreement is a bias-augmenting noise term in the variance forecast of the LP
- While some assumptions appear strict, they do not appear irrelevant empirically
Dawid-Sebastiani score (DSS) measures accuracy of density forecasts based on forecasts of mean and variance only. DSS is given by:

\[
DSS = \frac{1}{2} \left( \ln (2\pi) + \ln V_k + \frac{(Y - M_k)^2}{V_k} \right)
\]

DSS equivalent to logarithmic score if forecast density is normal.

Based on some information set \(\mathcal{I}\), DSS is minimized by:

1. forecasting \(M_k\) which minimizes \(\mathbb{E} \left[ (Y - M_k)^2 \middle| \mathcal{I} \right]\)
2. forecasting \(V_k = \mathbb{E} \left[ (Y - M_k)^2 \middle| \mathcal{I} \right]\)

In the context of linear pooling and unbiased \(V_i\)'s, this is equivalent to:

1. using Bates-Granger weights \(\omega^*\) for \(M_c\)
2. using \(V_{VULP}\)
Monte Carlo simulations - Set-up

Target variable given by

\[ Y = X_1 + X_2 + U \]

with

\[
\begin{bmatrix}
X_1 \\
X_2 \\
U
\end{bmatrix}
\sim
\left(\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 \\
0 & \sigma^2_{X_2} & 0 \\
0 & 0 & 1
\end{bmatrix}\right)
\]

Individual mean and variance forecasts are

\[ M_i = X_i \quad i = 1, 2 \]

\[ V_1 = 1 + \sigma^2_{X_2} \]

\[ V_2 = 2 \]

and combined mean forecast is

\[ M_c = \omega_1 M_1 + (1 - \omega_1) M_2 \]
Monte Carlo simulations - Set-up

- Densities of LP, CLP and VULP are given by

\[
\begin{align*}
 f_{LP} &= \omega_1 f_1 (M_1, V_1) + (1 - \omega_1) f_2 (M_2, V_2) \\
 f_{CLP} &= \omega_1 f_1 (M_c, V_1) + (1 - \omega_1) f_2 (M_c, V_2) \\
 f_{VULP} &= \omega_1 f_1 (M_c, V_1 - \mathbb{E} [D]) + (1 - \omega_1) f_2 (M_c, V_2 - \mathbb{E} [D])
\end{align*}
\]

implying variance forecasts

\[
V_{LP} = \omega' V + D, \quad V_{CLP} = \omega' V, \quad V_{VULP} = \omega' V - \mathbb{E} [D]
\]

- Two cases considered
  1. \(X_1, X_2\): normally distributed, \(\sigma_{X_2}^2 = 1.5\)
     \(f_1, f_2\): normal densities, variance-unbiased
  2. \(X_1, X_2\): scaled \(t\)-distributed (5 df), \(\sigma_{X_2}^2 = 1\)
     \(f_1, f_2\): scaled \(t\)-distributed (5 df) densities, variance-unbiased

- \(U\) normally distributed in both cases
Monte Carlo simulations - Expected variance

Variance, Case 1

Variance, Case 2

Combination Weight $\omega_1$

Variance

LP CLP VULP

Correlation $D, S$

Correlation $D, S$

Figure 1: Left column: Simulation results for case 1 (Gaussian forecasts, $\sigma^2 X^2 = 1.5$). Right column: Simulation results for case 2 (rescaled $t$-distributed density forecasts, $\sigma^2 X^2 = 1$). The first row shows the forecast variance of the linear pool (LP), the centered linear pool (CLP) and the variance-unbiased linear pool (VULP), plotted against $\omega_1$. The second row shows the correlation between disagreement $D$ and the squared forecast error $S$ of the combined mean forecast, again plotted against $\omega_1$. The vertical blue line indicates the MSFE-optimal weight $\omega^*_1$. 

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Monte Carlo simulations - Correlation $D, S$

Figure 1: Left column: Simulation results for case 1 (Gaussian forecasts, $\sigma^2_X = 1.5$). Right column: Simulation results for case 2 (rescaled $t$-distributed density forecasts, $\sigma^2_X = 1$). The first row shows the forecast variance of the linear pool (LP), the centered linear pool (CLP) and the variance-unbiased linear pool (VULP), plotted against $\omega_1$. The second row shows the correlation between disagreement $D$ and the squared forecast error $S$ of the combined mean forecast, again plotted against $\omega_1$. The vertical blue line indicates the MSFE-optimal weight $\omega^*_1$. 

Variance, Case 1

<table>
<thead>
<tr>
<th>Combination Weight $\omega_1$</th>
<th>LP Variance</th>
<th>CLP Variance</th>
<th>VULP Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>2.0</td>
<td>2.5</td>
<td>2.0</td>
</tr>
<tr>
<td>0.25</td>
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<td>2.0</td>
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<td>0.50</td>
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<td>2.5</td>
<td>2.0</td>
</tr>
<tr>
<td>0.75</td>
<td>2.0</td>
<td>2.5</td>
<td>2.0</td>
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<tr>
<td>1.00</td>
<td>2.0</td>
<td>2.5</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Correlation $\text{Cor}(D, S)$, Case 1

Correlation $\text{Cor}(D, S)$, Case 2
Figure 2: Left column: Simulation results for case 1 (Gaussian forecasts, $\sigma^2 = 1.5$). Right column: Simulation results for case 2 (rescaled $t$-distributed density forecasts, $\sigma^2 = 1$).

The first row shows the Dawid-Sebastiani score for the linear pool (LP), the centered linear pool (CLP) and the variance-unbiased linear pool (VULP), plotted against the combination weight $\omega_1$. A lower score indicates a more accurate forecast. The vertical blue line indicates the MSFE-optimal weight $\omega_1^*$. The second row shows analogous plots for the log score.
Monte Carlo simulations - Log score

Figure 2: Left column: Simulation results for case 1 (Gaussian forecasts, $\sigma^2 = 1.5$). Right column: Simulation results for case 2 (rescaled $t$-distributed density forecasts, $\sigma^2 = 1$).

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Empirical Application - Inflation forecasts

- Predict distribution of change in quarterly US GDP deflator with
  - **UCSV** model (Stock&Watson 2007, Chan 2013)
  - **CMM** model (Clark, McCracken&Mertens 2020)
  uses SPF point forecasts + Bayesian stochastic volatility model for the
  conditional distribution of SPF point forecast errors
- Forecasts for 1-5 quarters ahead, evaluation sample 1976-2018
- Combine forecasts with **LP** and **CLP**
- $\omega_1$ is weight of **CMM** model
Models’ mean and variance forecasts

Mean, $h = 1$

Mean, $h = 5$

Variance, $h = 1$

Variance, $h = 5$

Figure 3: Mean (top row) and variance (bottom row) of the forecast distributions for the CMM and UCSV model. Left and right column correspond to shortest and longest forecast horizon ($h = 1$ and $h = 5$). Evaluation sample ranges from 1976:Q2 to 2018:Q3.

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Pools’ variances and their correlations with squared errors

Variance, $h = 1$

Variance, $h = 5$

$\text{Cor}(V, S), h = 1$

$\text{Cor}(V, S), h = 5$

Figure 5: First row: Dawid-Sebastiani score, plotted against $\omega_1$, the combination weight of the CMM model. Second row: Log score plotted against $\omega_1$. Left and right column correspond to shortest and longest forecast horizon ($h = 1$ and $h = 5$). A lower score indicates a more accurate forecast. MSFE-optimal weight is marked by blue vertical line in each plot. Evaluation sample ranges from 1976:Q2 to 2018:Q3.
Tests of predictive accuracy

- $t$-statistics of Diebold-Mariano tests, LP vs CLP
  - density combinations use equal weights ($\omega_1 = 0.5$)
  - negative values indicate superiority of LP

<table>
<thead>
<tr>
<th>$h$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$DSS$</td>
<td>-0.20</td>
<td>2.55</td>
<td>3.84</td>
<td>4.30</td>
<td>3.23</td>
</tr>
<tr>
<td>log score</td>
<td>-0.81</td>
<td>1.10</td>
<td>3.30</td>
<td>2.62</td>
<td>2.30</td>
</tr>
</tbody>
</table>

- LP better at $h = 1$
- CLP better at $h > 1$, mostly significant
Additional pools

- Gneiting & Ranjan (2013) suggest adjusted LPs that require a training sample
  - spread-adjusted linear pool
    \[ f_{sLP}(Y) = \omega_1 f_1(Y; M_1, \kappa \times V_1) + (1 - \omega_1) f_2(Y; M_2, \kappa \times V_2) \]
  - beta-transformed linear pool
    \[ f_{bLP}(Y) = f_{LP}(Y) \times b_{\alpha, \beta}(F_{LP}(Y)) \]
    with \( \kappa \) and \( \alpha, \beta \) chosen to maximize fit in training sample, \( b_{\alpha, \beta} \) pdf of beta distribution, \( F_{LP} \) cdf of LP

- Both adjustments can also be applied to CLP
  - spread-adjusted centered linear pool
    \[ f_{sCLP}(Y) = \omega_1 f_1(Y; M_c, \kappa \times V_1) + (1 - \omega_1) f_2(Y; M_c, \kappa \times V_2) \]
  - beta-transformed centered linear pool
    \[ f_{bCLP}(Y) = f_{CLP}(Y) \times b_{\alpha, \beta}(F_{CLP}(Y)) \]
Scores for all pools, $\omega_1 = 0.5$

- Best results (over all pools) bold
- Better results (pairwise comparison) in italics

<table>
<thead>
<tr>
<th></th>
<th>$DSS$</th>
<th>$\log$ score</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>LP</td>
<td>1.40</td>
<td>1.52</td>
</tr>
<tr>
<td>CLP</td>
<td>1.40</td>
<td>1.51</td>
</tr>
<tr>
<td>sLP</td>
<td>1.40</td>
<td>1.51</td>
</tr>
<tr>
<td>sCLP</td>
<td>1.41</td>
<td><strong>1.51</strong></td>
</tr>
<tr>
<td>bLP</td>
<td>1.42</td>
<td>1.52</td>
</tr>
<tr>
<td>bCLP</td>
<td>1.43</td>
<td>1.52</td>
</tr>
</tbody>
</table>

- Centered pools tend to be more accurate
- Spread-adjusted centered linear pool often works best
CLP gives less biased variance forecasts than LP if individual densities are variance-unbiased.

CLP is LP without disagreement - very easy to implement.

Under certain conditions, disagreement contains no additional information about forecast uncertainty.

In empirical application to inflation forecasts, CLP tends to outperform LP.

→ When pooling density forecasts with reasonable variance forecasts, it’s worth trying out the CLP.